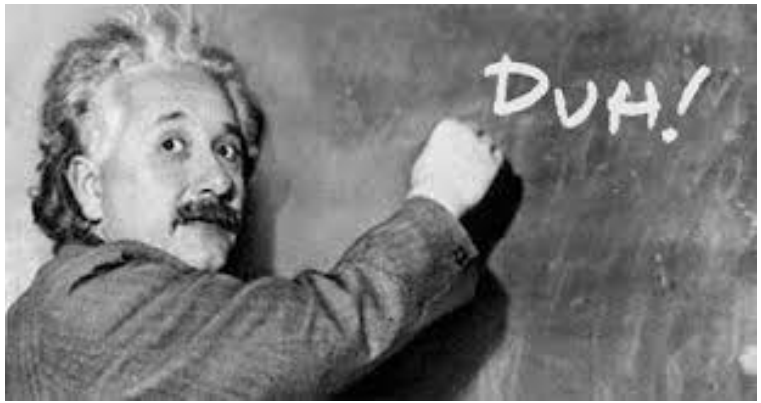
Summations in Probability

Lecture by Mark Lee

The Problem

Q: What is the average number of six-sided dice rolls needed to get a 6?
(Fifty Challenging Problems in Probability, Question 4)

A: 6....



Q: *Why?*

Binomial Distributions

$$P(X) = {}_nC_x \, p^x (1-p)^{n-x}$$

- Probability distribution of repeated trials - formula & symbol (right)
- Variables:
  - x → # of Successes
  - n → # of Trials
  - P(X) → probability of X successes in N trials
  - p → probability of success for each trial
- Examples: Dice rolling, sampling (w/ **replacement!!**)

Bin (n,p)

The Binomial Theorem

- Application to Algebra
  - Expansion of a binomial (two-variable equation)
- Backwards Usage
- Inductive Proof

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

A Slightly Different Scenario: Expected Value

- Expected Value of a Binomial distribution B (n, p)
  - Expected Value: Sum of all outputs x probability
- Motif: Removing a factor of K!

Conclusion: E(X) = np!

Sample Problems 1/2

1. John guesses on all 30 problems in a 5-answer multiple choice quiz. What is the probability that he will have gotten exactly 10 problems right?
2. Felicity rolls a 6-sided die 5 times. Calculate the probability that she will have rolled at least 2 ones during this die roll sequence.

Answers

1. 0.0355
2. 0.9966

Infinite Geometric Summations

- Geometric Series
  - Common Ratio: r
  - First Term: a
- Geometric Summation Formula
  - Proof
- Infinite Summation
  - Prerequisites
  - N → ∞!
  - Involves usage of limits

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_\infty = \frac{a_1}{1 - r}$$

Sample Problems 3/4

1. A patient receives repeated doses of a drug. Each time the drug is administered, there is a 5% chance the patient will be cured. What is the probability that the patient will be cured within 20 successive doses?
2. A, B and C are to fight a three-cornered pistol duel. All know that should each player aim to kill, A's chance of hitting his target is 0.3, C's is 0.5, and B never misses. The three duelists will fire at their choice of target in succession cyclically in alphabetic order until only one man is left standing. What should A's strategy be?

Answers

3. 0.641514
4. Deliberate miss followed by firing at B

The Proof

- Setting Up
- “Arithmetico-Geometric Sum”
- Mark’s Original Method
  - “Grid of Infinity” - Double Infinite Sum!
- The Approved Method
  - Classic Geometric Summation Trick
  - Inductive Deduction

Generalization

- Substitute Variables!
  - Number of Trials: 6 → N
  - Probability of Success: ⅙ → P
  - Probability of Failure: ⅚ → Q
  - Outcome: “6” → Event “A”
- Implications

Theorem: **Given repeated random trials of a binomial sample space where outcome A has probability $P > 0$, NP = 1 where N is the expected number of trials required to yield the first instance of A.**

Extension II: Lucky Cereal

1. Coupons in cereal boxes are numbered from 1 to 5, and a set of one of each is required for a prize. With one coupon per box (and an infinite number of boxes available), how many boxes on average are required to make a complete set? *(Fifty Challenging Problems in Probability, Question 14)*

Answer: ≅11.42 boxes

*Generalize to all coupons from 1 to N! (Harmonic Sum)

A Different Scenario: Deck of Cards

1. A standard deck of cards has 52 cards, 4 of which are aces. On average, how many cards must be drawn off the top of a standard deck of cards before the first ace is drawn? (*NOTE: This total includes the first ace!)

Answer: 10.6 cards

Conclusion

Thank You For Listening!